Greedy Mesh Coarsening with Spectral Guarantees

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Goals

Overall Goal: Develop a "spectrally aware" graph coarsening method which preserves eigen-properties of the normalized graph Laplacian. Additionally, compare this with existing coarsening methods.

Why?

- Eigenvectors and eigenvalues of the Laplacian convey geometric information about the graph including cut values. [3]
- Multilevel partitioning methods such as METIS [7, 6] often use spectral cuts at the smallest level of coarsening.

Coarsening

Graph Coarsening: Find a representation of an original graph $G = (V, E)$ with $|V| = n$, with a reduced dimension $G_c = (V_c, E_c)$ with $|V_c| = n_c < n$.

There are many methods to perform this task.

Method	Abbreviation	Objective
Heavy Weight [1]	HWC	$\max_{(u,v)\in E} \left\{ \min \left\{ \frac{W_{uv}}{d_u}, \frac{W_{uv}}{d_v} \right\} \right\}$
Algebraic Distance [2]	ADC	$\min_{(u,v)\in E} \left\{ \|x_u^k - x_v^k\|_\infty \right\}$
Adjacency [5]	ASC	$\min_{(u,v)\in E} \left\{ \left\| \frac{W_{u:}}{d_u} - \frac{W_{v:}}{d_v} \right\|_1 \right\}$
Greedy Fiedler [4]	GFSC	—
METIS [7]	METIS	—

Coarsening

Graph Coarsening: Find a representation of an original graph $G = (V, E)$ with $|V| = n$, with a reduced dimension $G_c = (V_c, E_c)$ with $|V_c| = n_c < n$.

With the exception of ASC, none of these methods provide explicit bounds for eigenvector or Eigenvalue approximation. For ASC specifically it has been proven that if $u, v \in V$ are merged such that $\left\| \frac{W_{u:}}{d_u} - \frac{W_{v:}}{d_v} \right\|_1 \leq \epsilon$, the eigenvalues of G_c approximate those of G to within ϵ.

$\Rightarrow$ **ASC closely preserves spectral cuts to wtihin some bound.**

An Observation

Nodal Domains are "easier" to compute than explicit eigenvectors. I.E. we can approximate nodal eigenvector embeddings without having to compute explicit eigenvectors.

- Spectral clustering is a particularly popular clustering technique in data mining, and its application to graph partitioning has uses in schemes like METIS.
- It is very expensive in general, $O(n^3)$. Can be done in $O(km)$ for sparse networks, but is sequential and may suffer from convergence issues.
- However, the sign patterns of these vectors can be computed in a far easier way.

An Observation

Nodal Domains are "easier" to compute than explicit eigenvectors. I.E. we can approximate nodal eigenvector embeddings without having to compute explicit eigenvectors.

Work done by Sahai et al. [8] and Zhu et al. [9] develops methods for distributed computation of local eigenvector components using the wave equation.

1. Iterate the wave equation on the graph.
   $x^k = 2x^{(k-1)} - x^{(t-2)} - c^2 \mathcal{L} x^{(k-1)}$
2. Then at each node, either take the Fourier transform, or perform the dynamic mode decomposition. Note that these can both be done embarrassingly in parallel.

Both of these methods require one large sequential step followed by a distributed step. This is in contrast to GMRES which requires $O(m)$ iterations for every eigenvector we wish to compute.

An idea

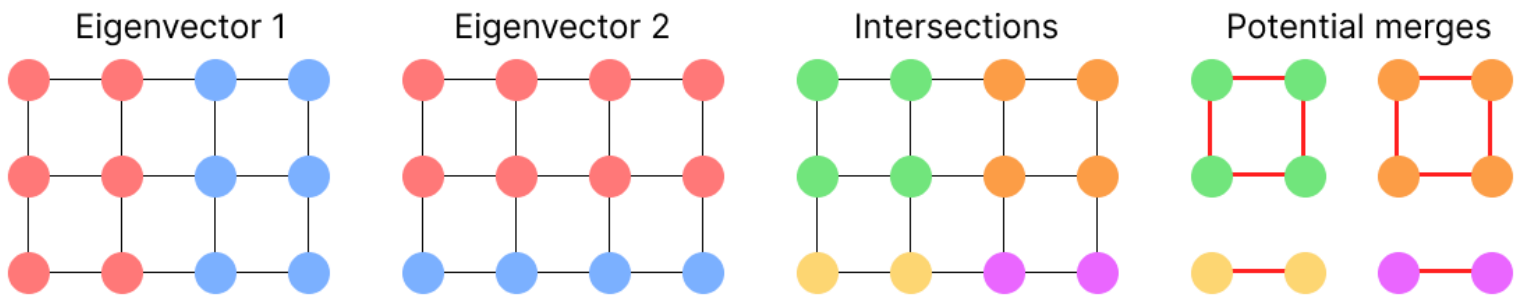
Use the nodal domains of eigenvectors to "filter" potential merge edges, and then run ASC coarsening on the remaining edges to the desired level of coarsening.

The coarsening method can be outlined as follows...

1. Compute the nodal domains of k eigenvectors on G, $\{x_i\}_{i\in[1..k]}$.
2. For each node $u \in V$, assign a vector of signs $\phi_u = [sign(x_0(u)), sign(x_1(u)), \cdots, sign(x_k(u))]$
3. Perform merges according to the ASC criteria between nodes $u, v \in V$ such that $\phi_u = \phi_v$.

NDC

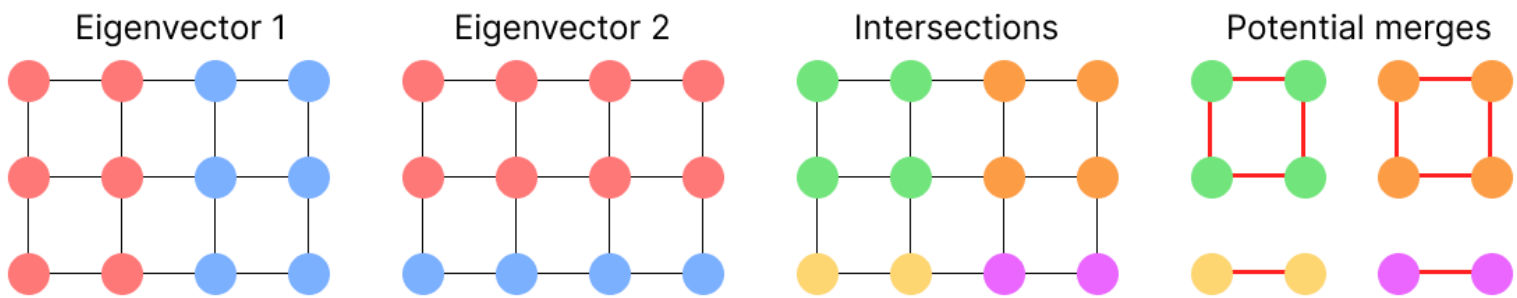
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NDC

Intuitively, the nodal domain coarsening method has some theoretical benefits.

- A spectral bound guaranteed by ASC.
- Any merged nodes have a similar spectral embeddings.
- The computation is distributable.



Results

We now compare these methods, and their spectral approximation properties on two suites of graphs. First, the **irregular dataset**, consisting of a mix of social networks, collaboration networks, road networks, email networks, and voting networks. Secondly the **mesh dataset** consisting of several mesh graphs.

Graph	Nodes	Edges	Type
CA-GrQc	5242	14496	Collaboration
CA-HepTh	9877	25998	Collaboration
email-Eu-core	986	332334	Email
facebook_combined	4039	88234	Social
lastfm_asia_edges	7624	27806	Social
Wiki-Vote	7115	103689	Voting
euroroad	1174	1417	Road
arenas-jazz	198	2277	Social
alligator	3208	9188	Mesh
cheburashka	6669	20001	Mesh
teapot	2259	6250	Mesh
woody	694	1960	Mesh
rocker-arm	10044	30132	Mesh

Eigenvector approximation

We first examine the eigenvector approximation properties of these methods. We do this by examining the inner product of their vector spaces.

$$\varepsilon(G_c) = \left\| \left(C^T V_c \right)^T V - I_k \right\|_F$$

"Lifting" Operator

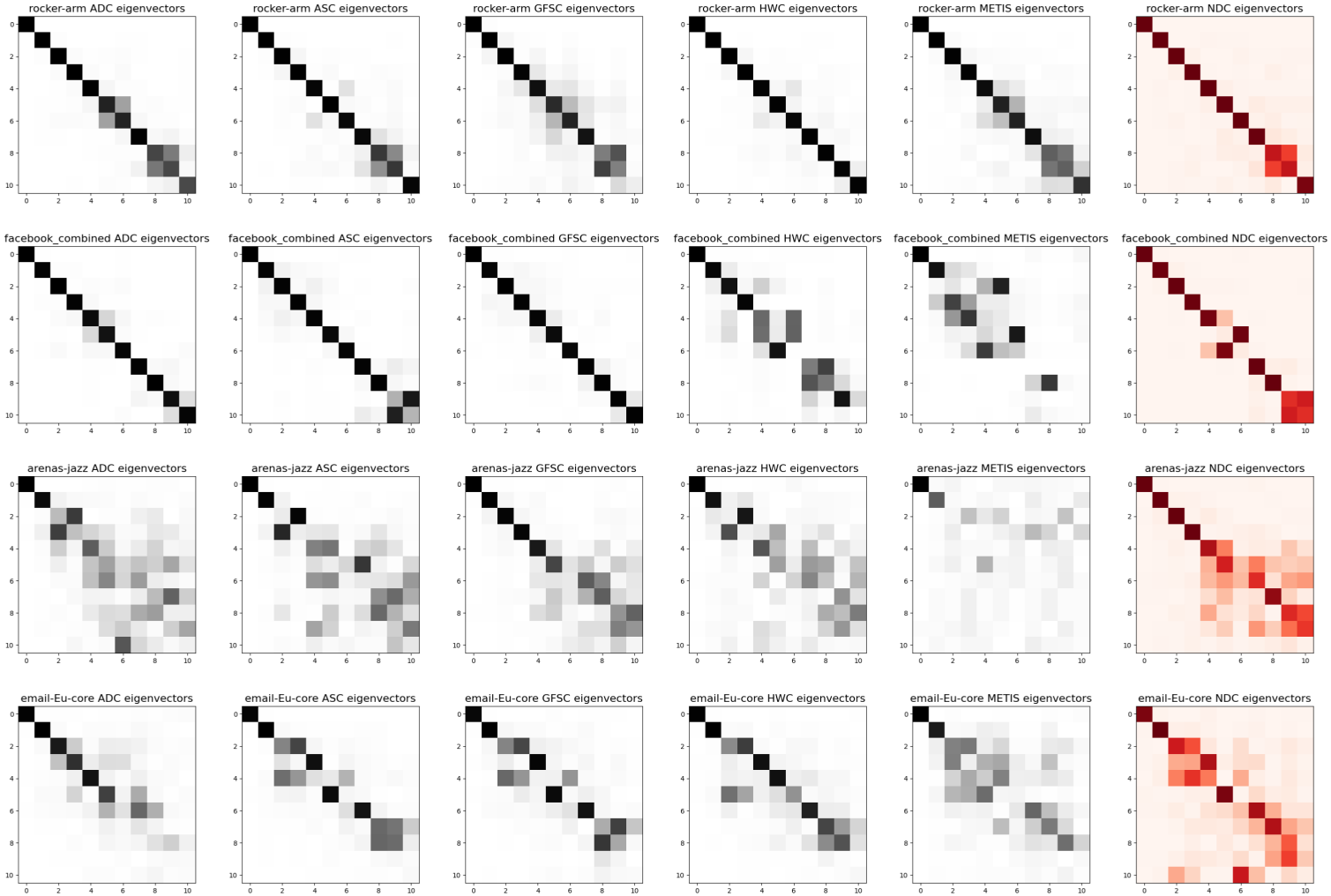
Coarse Eigenvectors

Original Eigenvectors

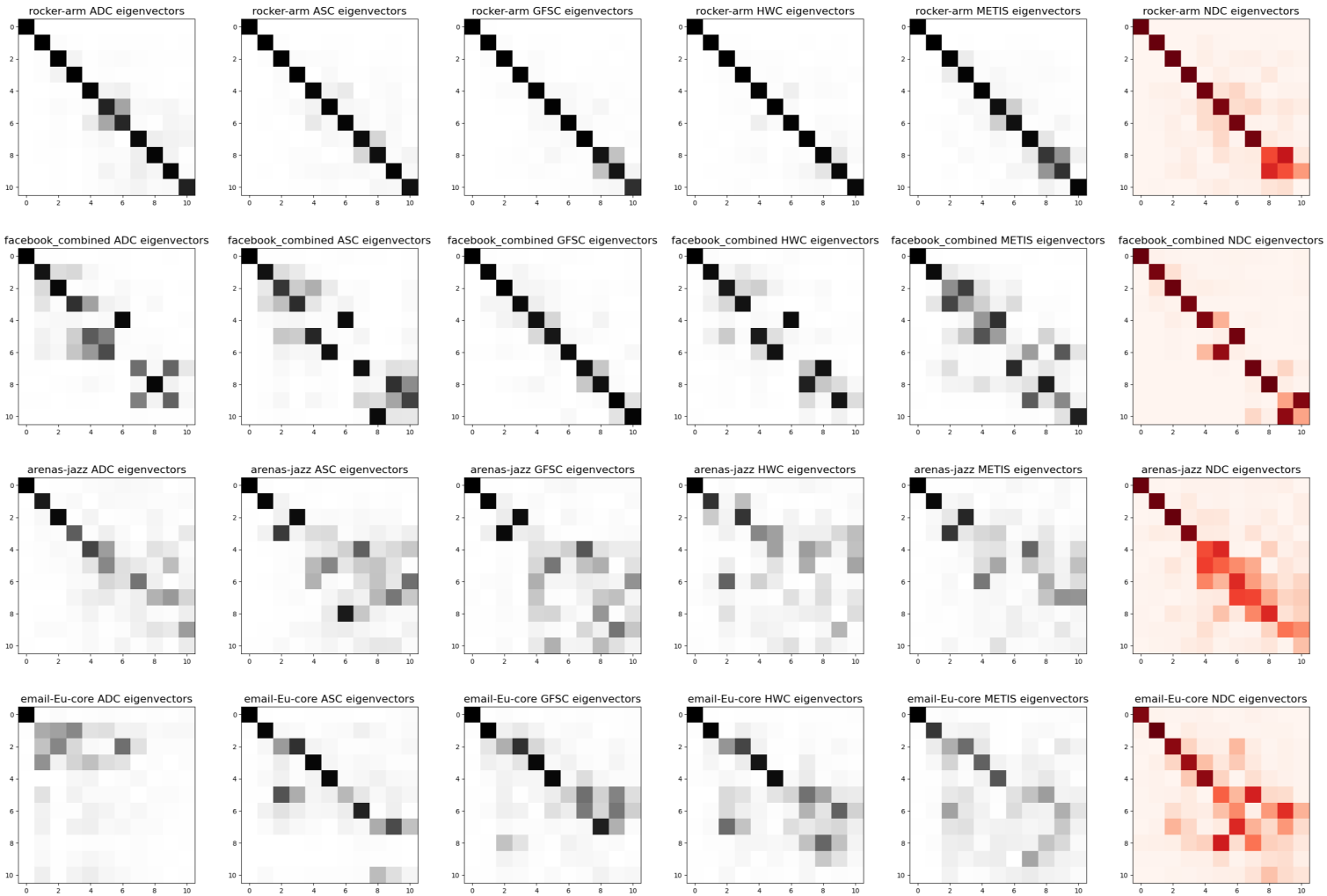
Identity Matrix

Intuitively, this captures the similarity in V_c and V by measuring the orthognality of their vectors. If orthognality is preserved, the inner product should be resemble a diagonal matrix, and this norm should be small.

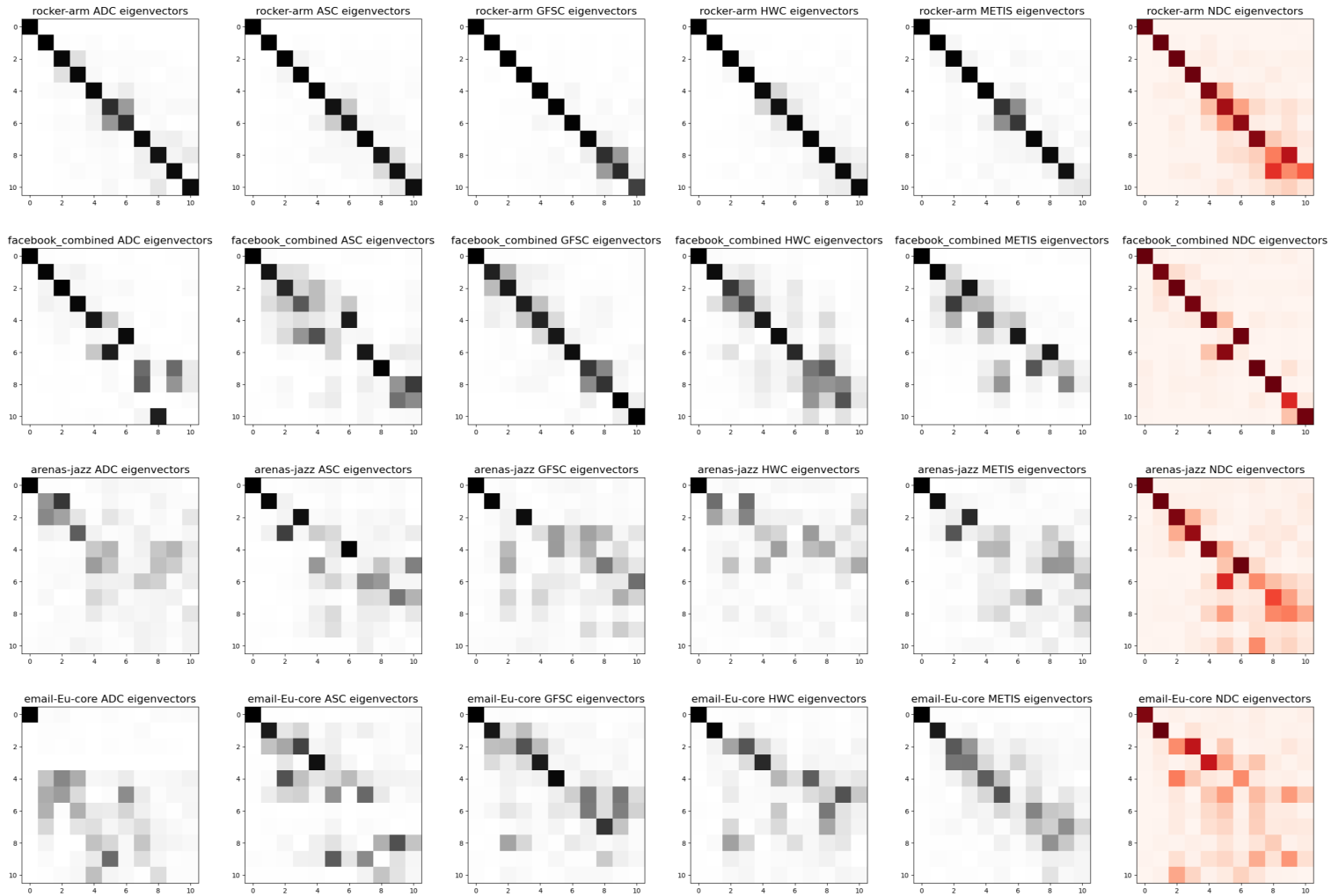
$\frac{n_c}{n} = 0.5$ coarsening eigenvector comparisons



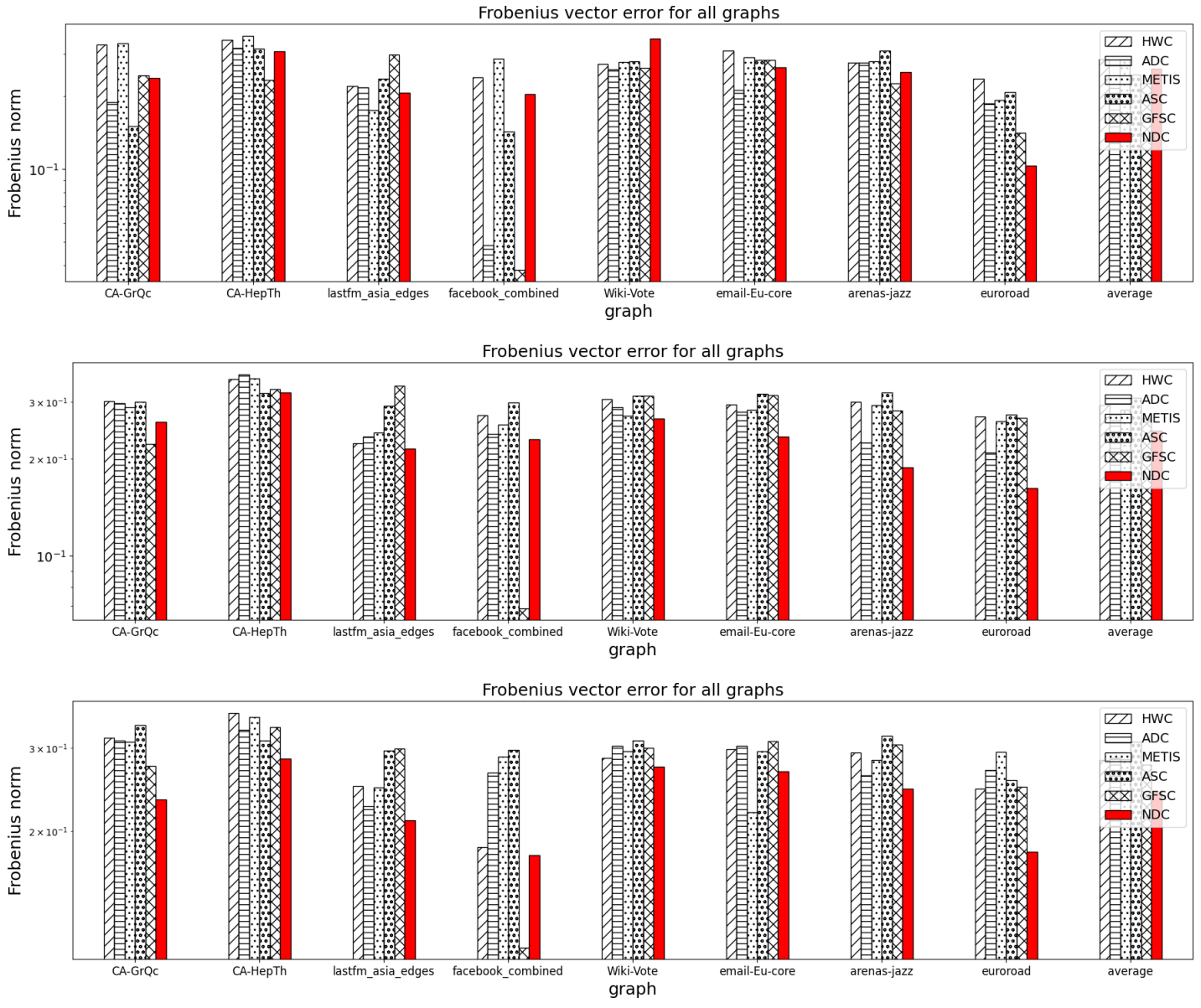
$\frac{n_c}{n} = 0.2$ coarsening eigenvector comparisons



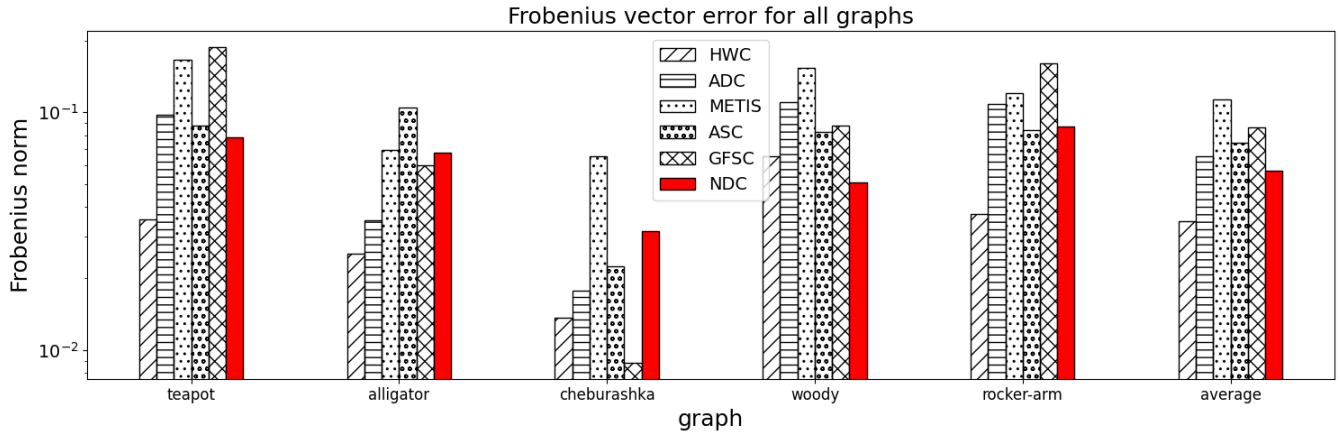
$\frac{n_c}{n} = 0.1$ coarsening eigenvector comparisons

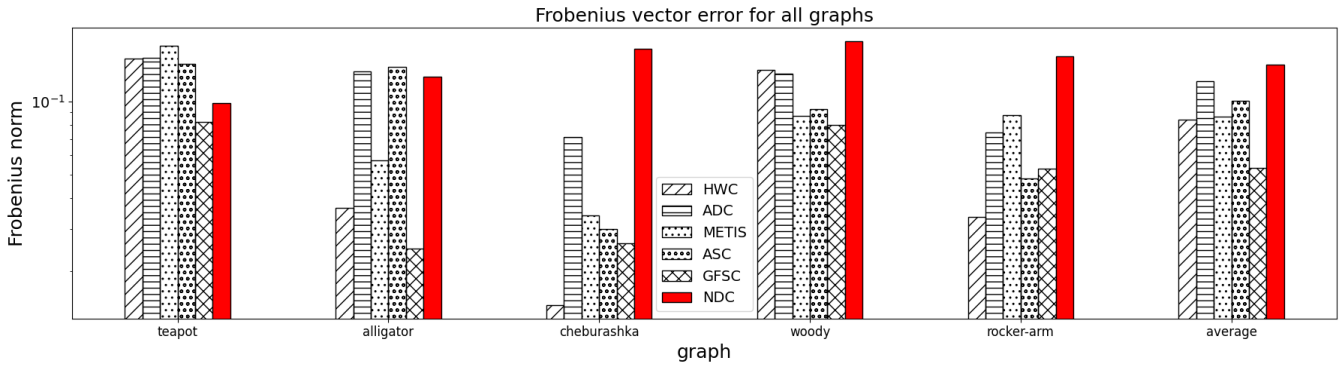


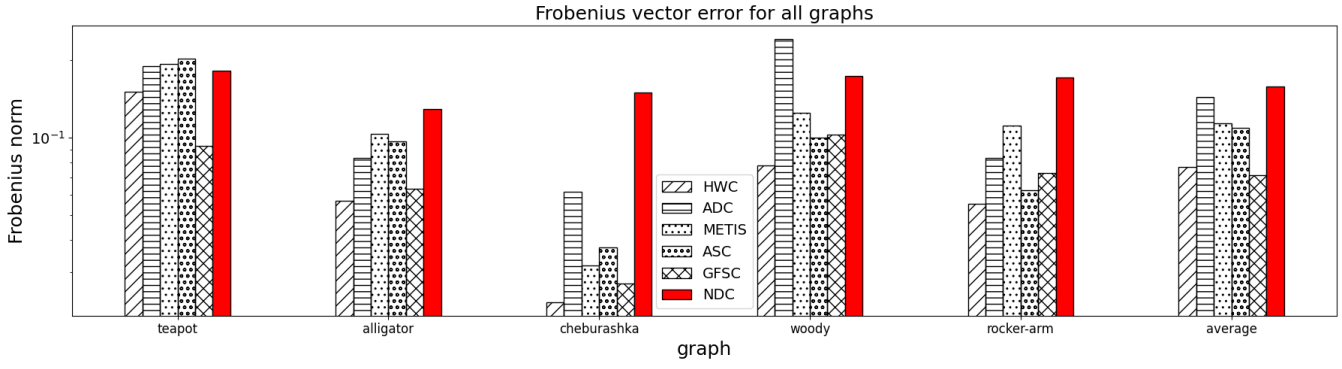
Eigenvector norm comparisons for **irregular dataset**.



Eigenvector norm comparisons for **mesh dataset**.







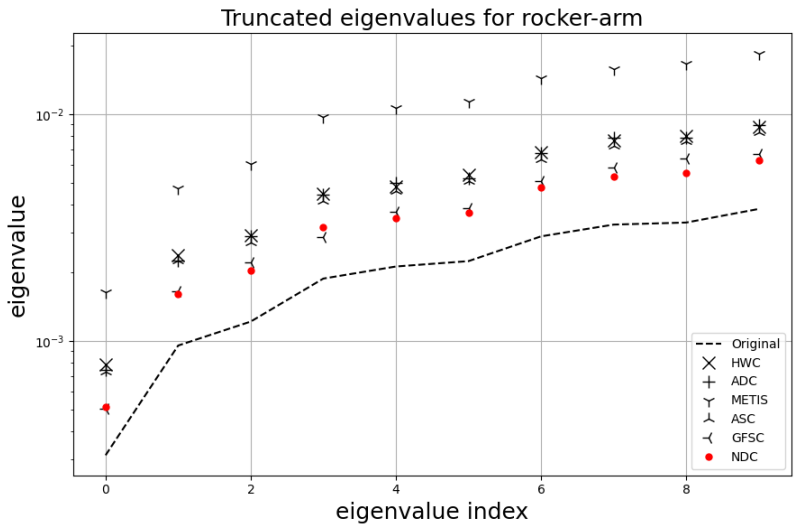
Eigenvalue approximation

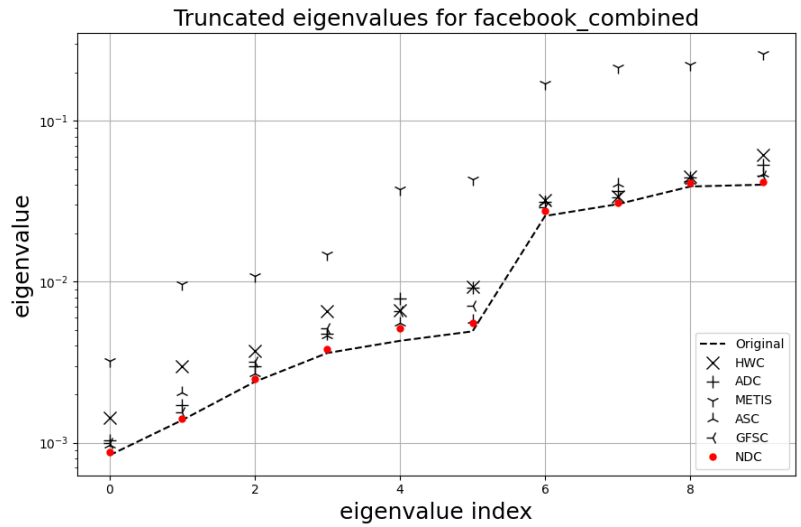
We now look at the eigenvalue approximation properties by examining the average difference in spectra between the coarsened graph and original graph.

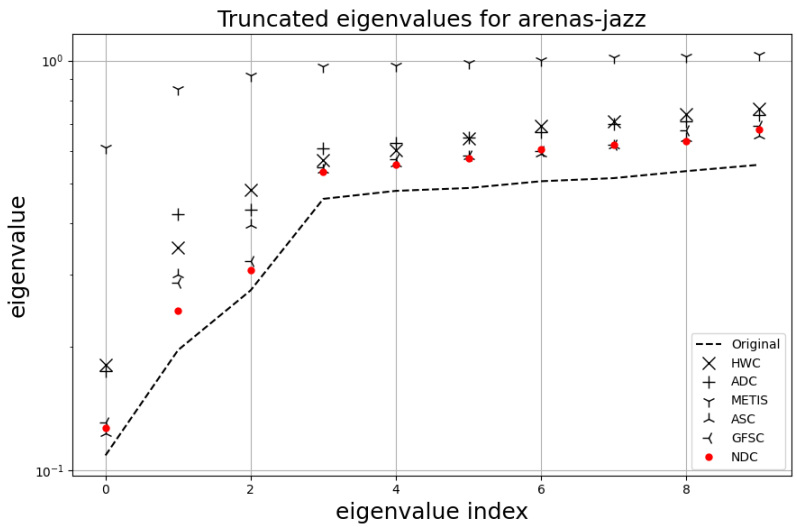
$$\varphi_k(G_c) = \frac{1}{k}\left(\sum_{i=0}^{k} |\lambda_i(G) - \lambda_i(G_c)|\right)$$

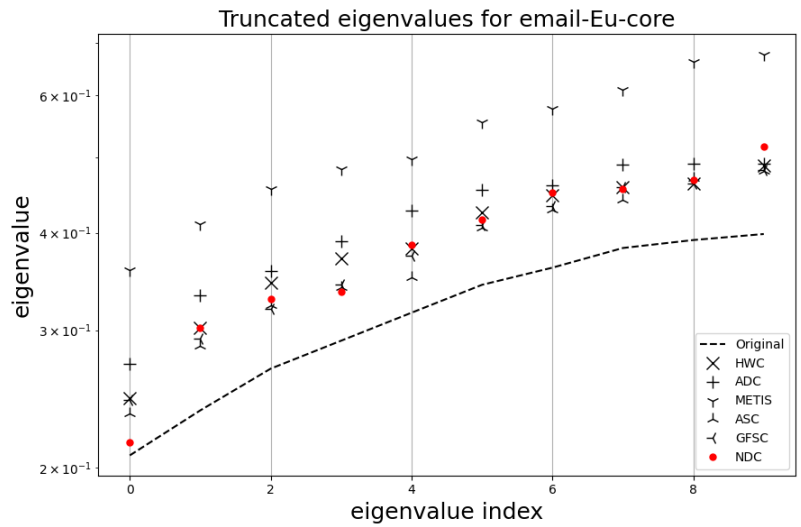
Intuitively this is measuring how "good" the spectral cuts of the coarse graph are in comparison to the spectral cuts of the original graph. Here we only condsider the k smallest eigenvalues because these control cut values. k is taken to be 10 for these experiments.

$\frac{n_c}{n} = 0.5$ coarsening eigenvalue comparisons

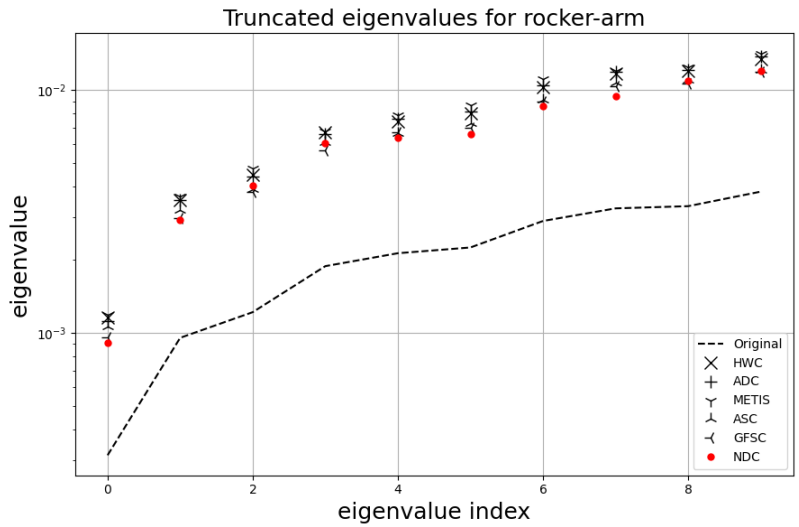


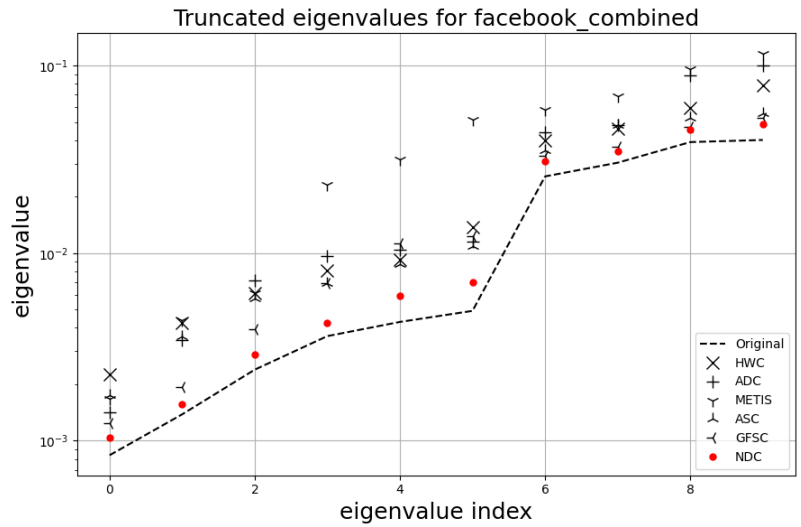


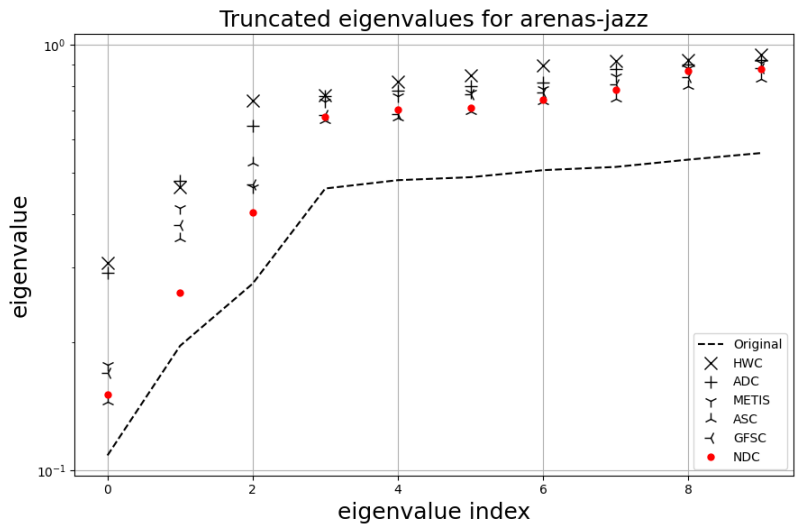


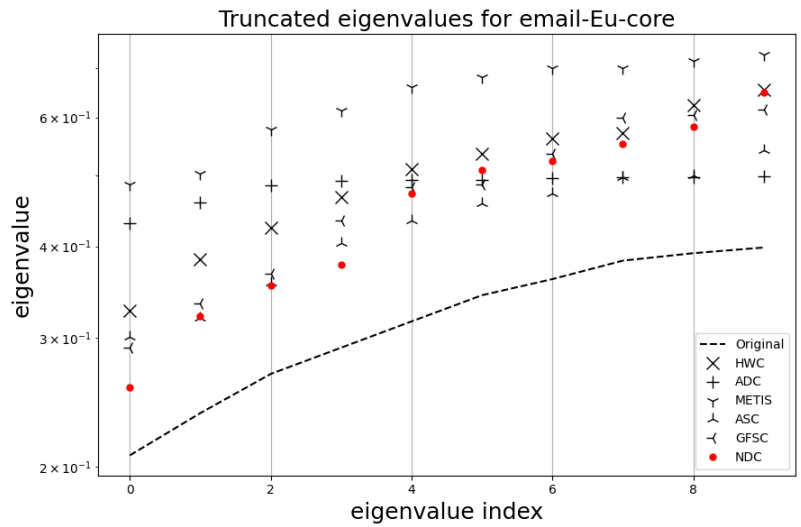


$\frac{n_c}{n} = 0.2$ coarsening eigenvalue comparisons

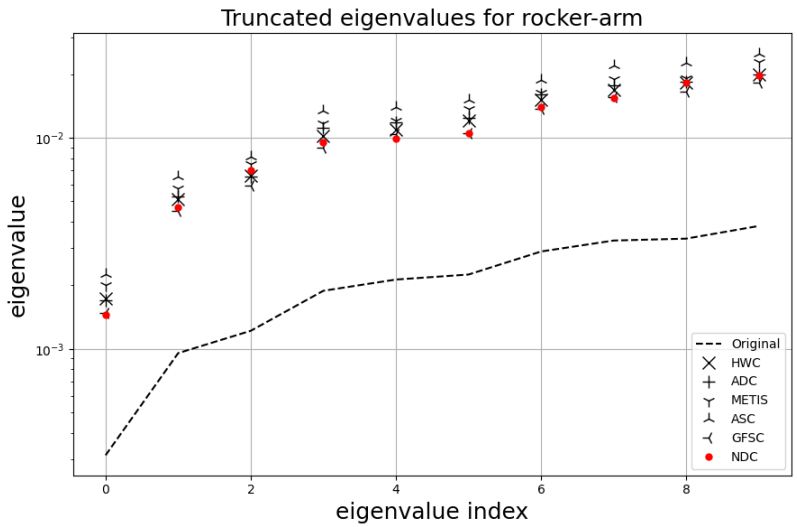


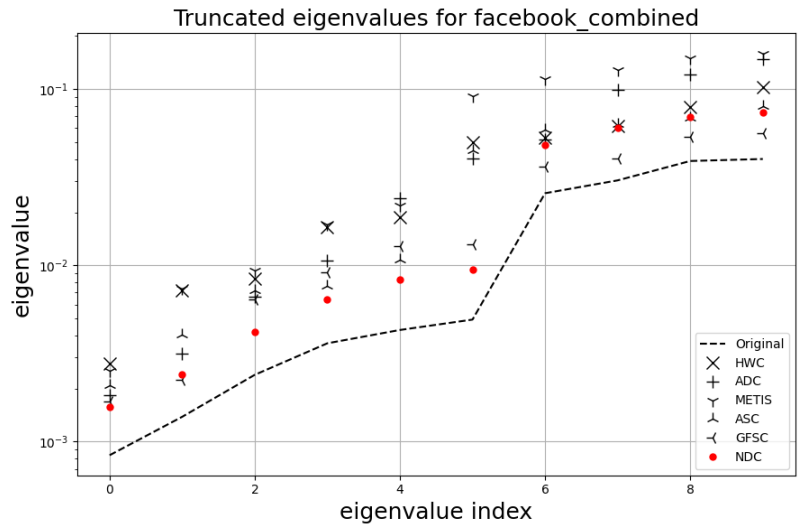


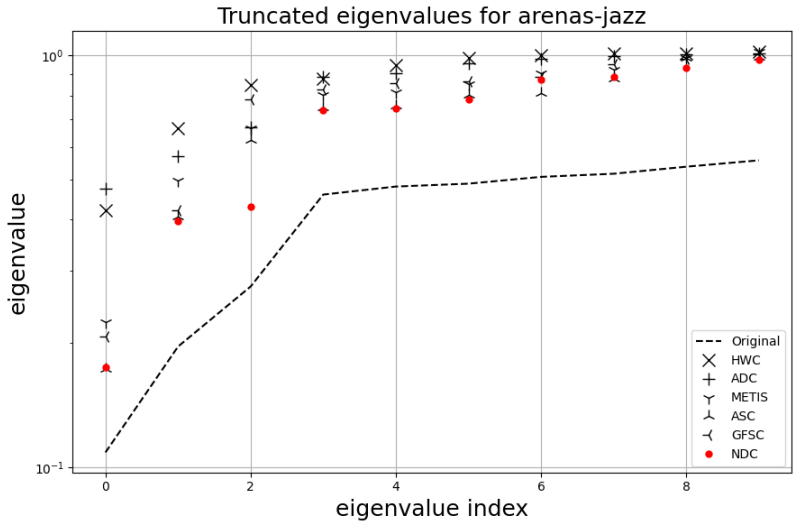


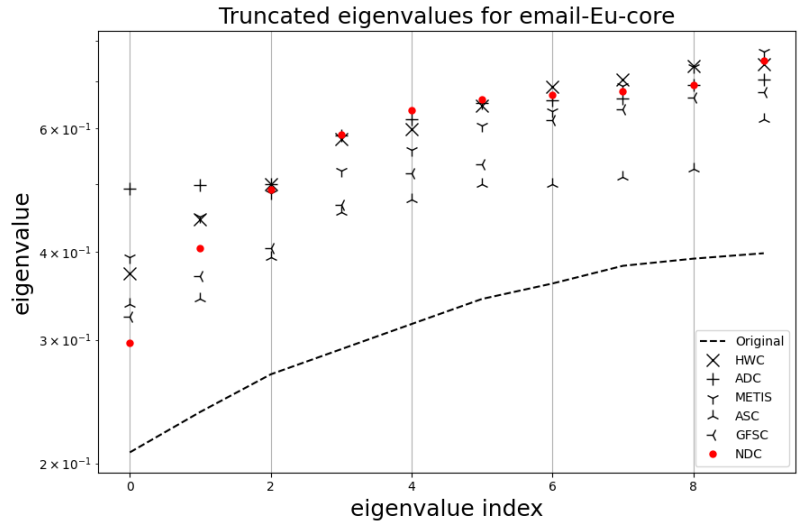


$\frac{n_c}{n} = 0.1$ coarsening eigenvalue comparisons

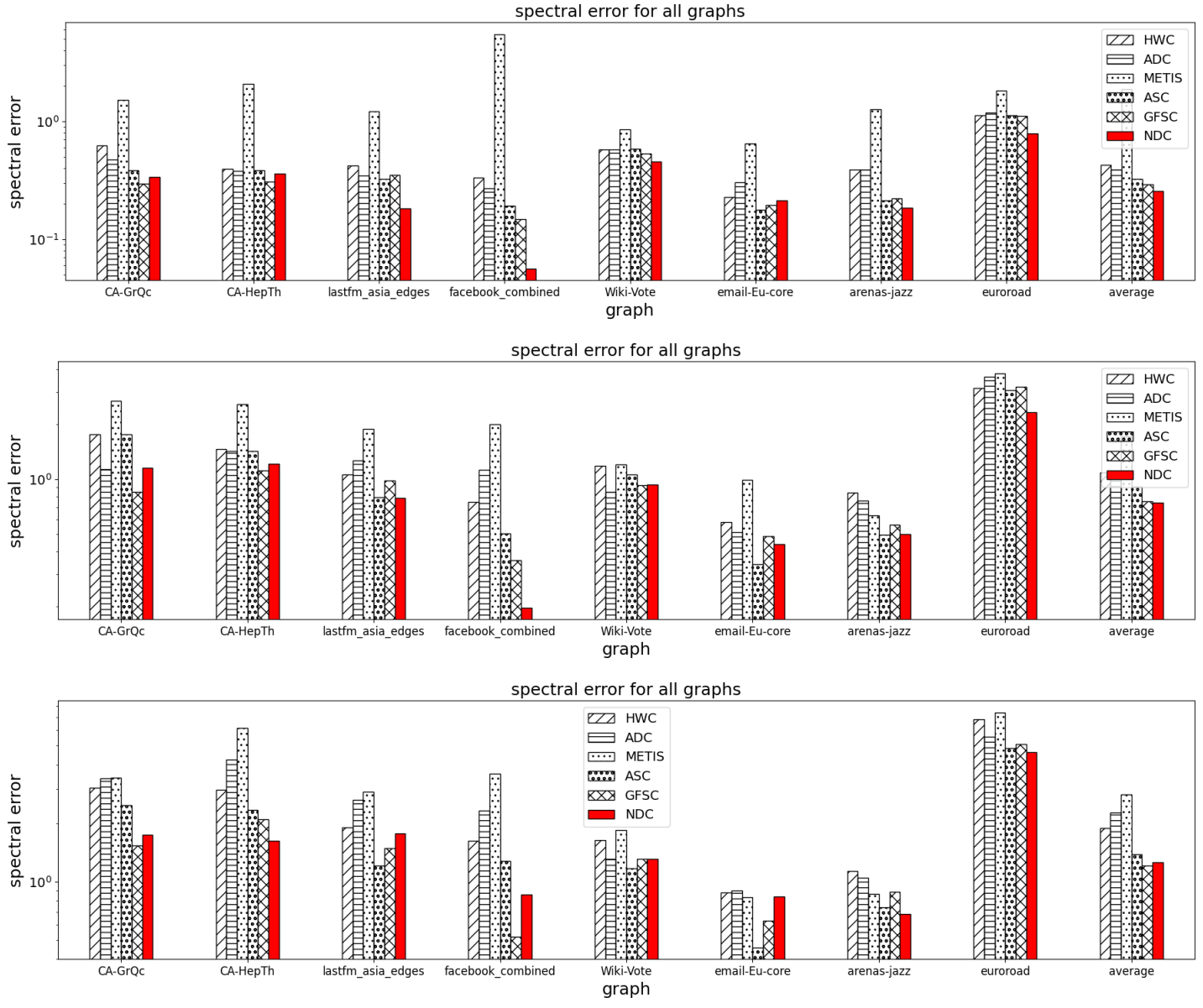




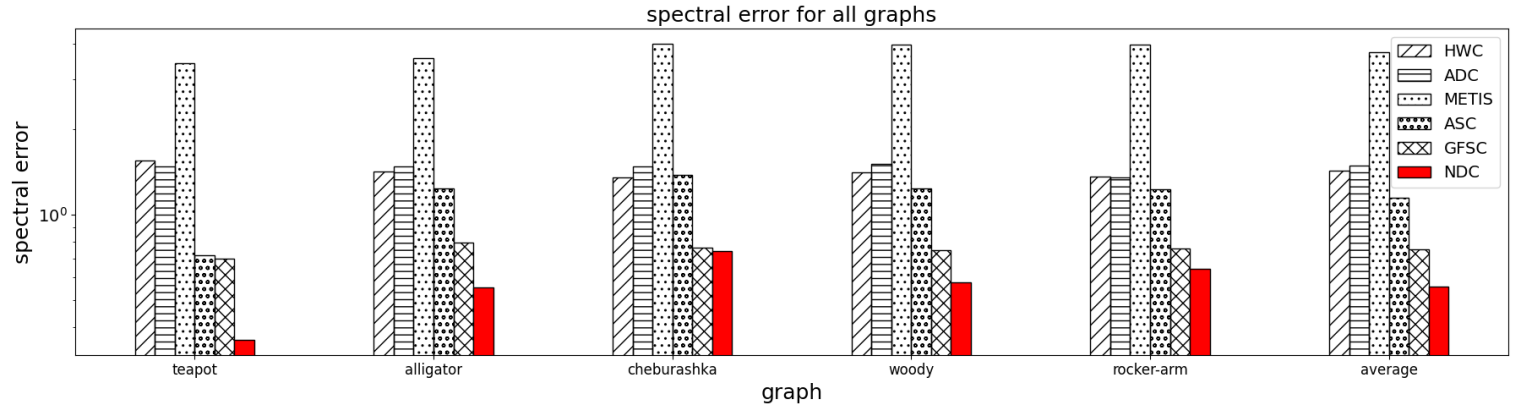


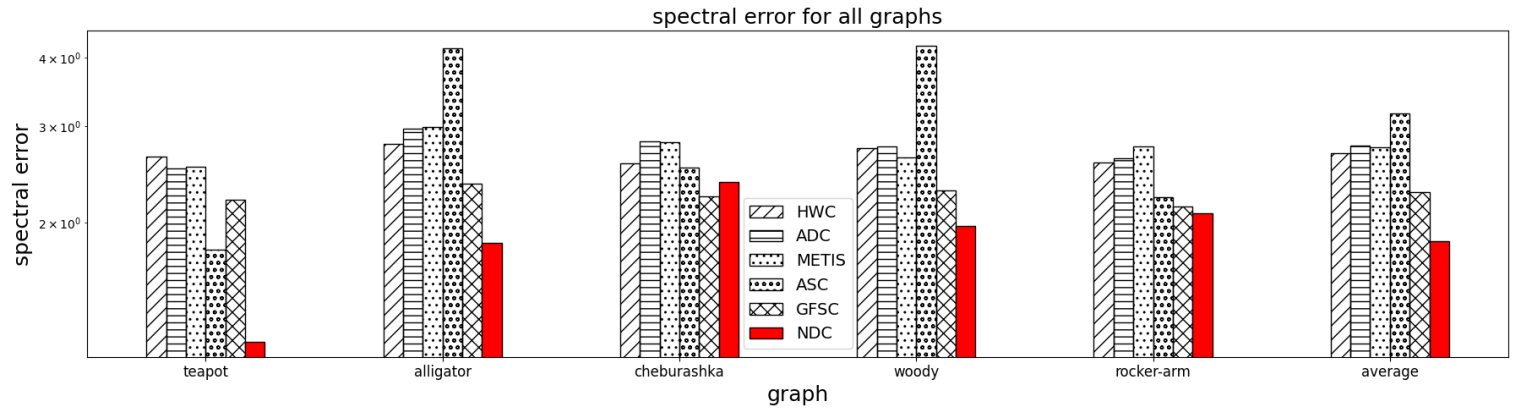


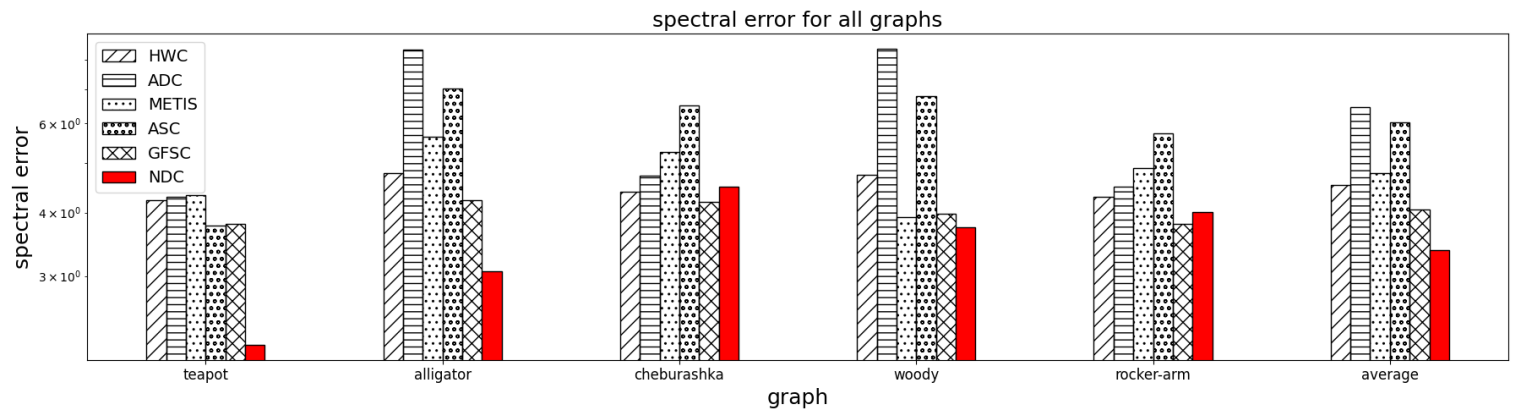
Eigenvalue comparisons for **irregular dataset**.



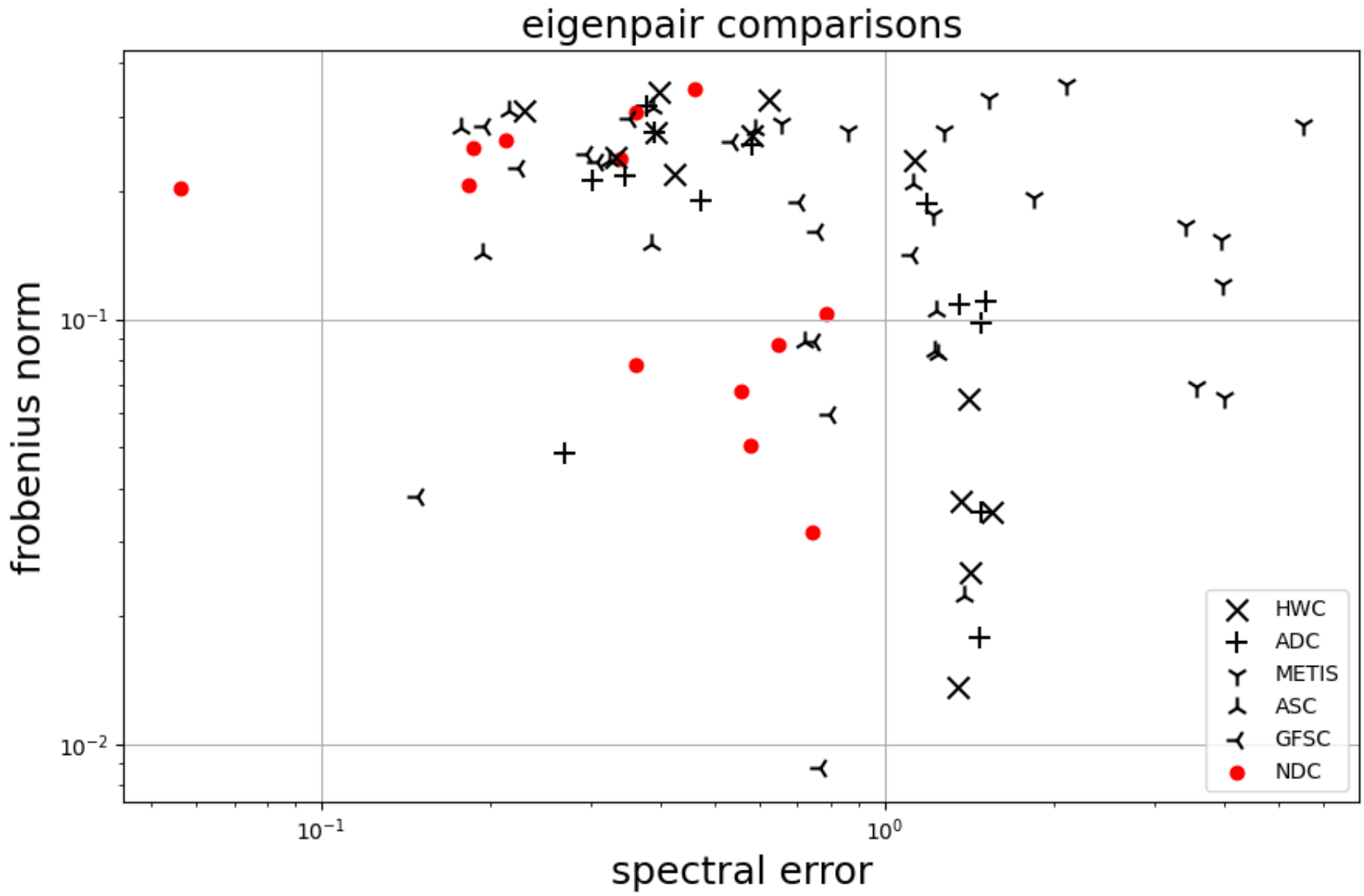
Eigenvalue comparisons for **mesh dataset**.



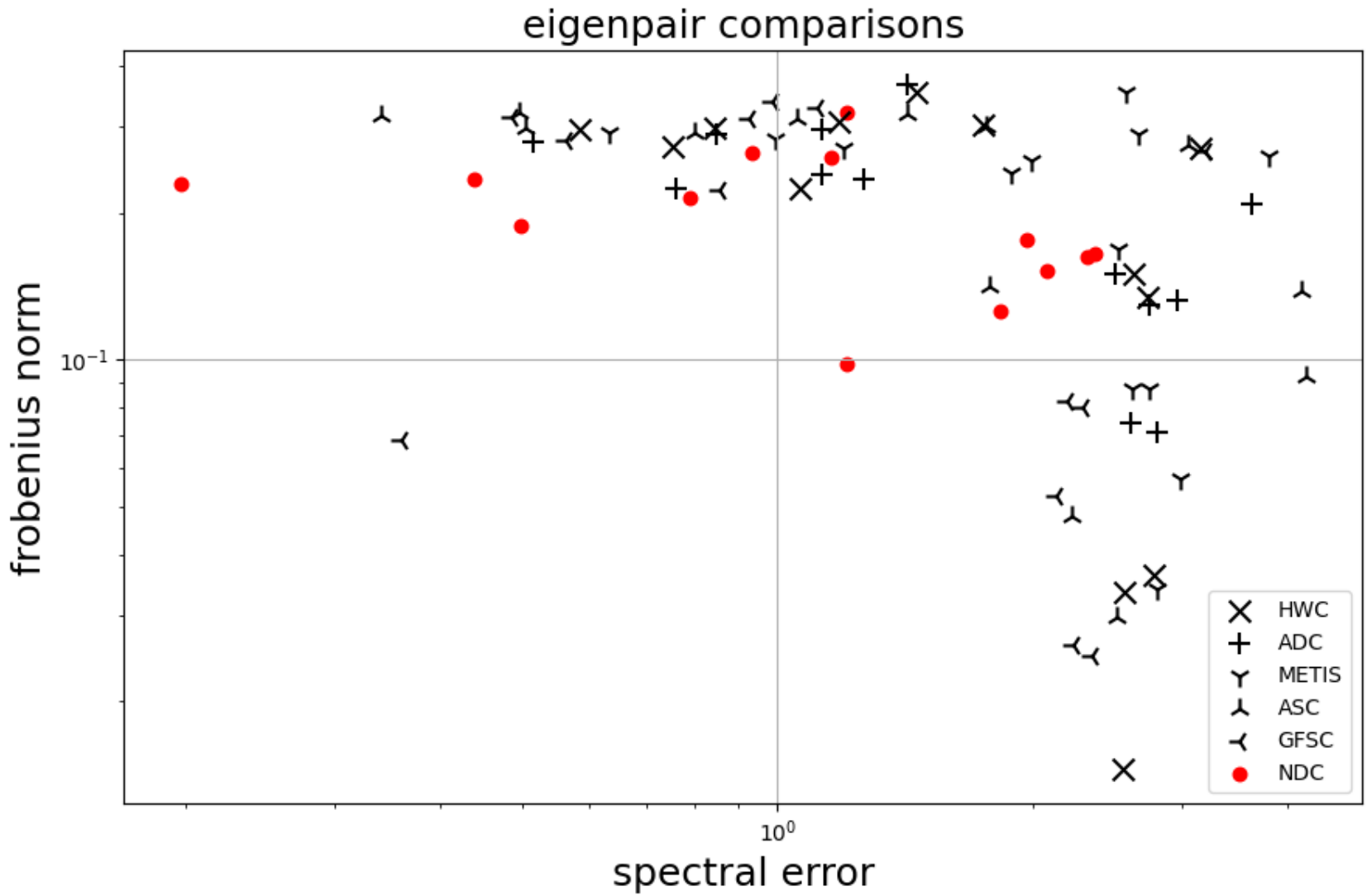




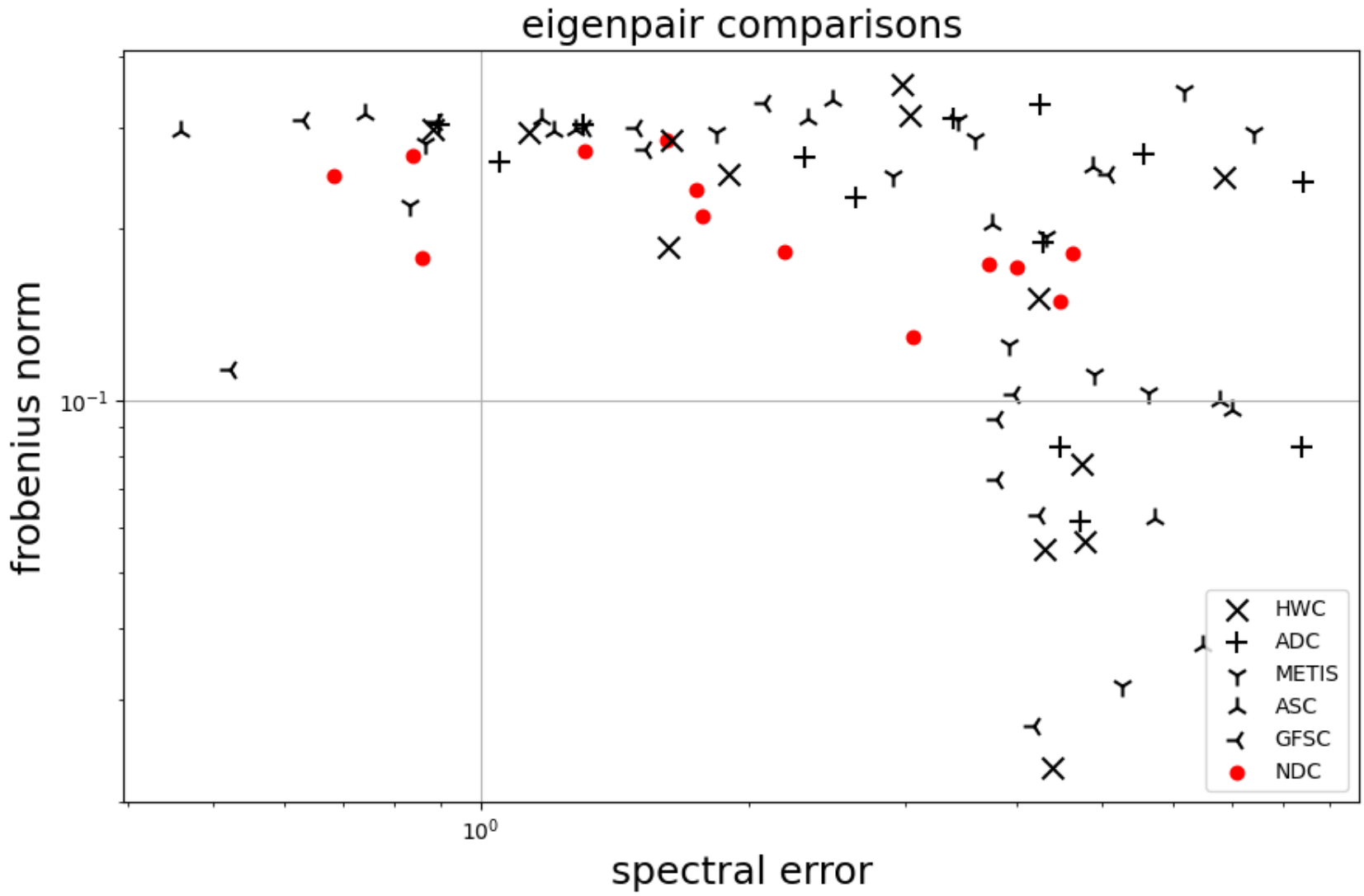
Vector versus value approximations for $\frac{n_c}{n} = 0.5$.



Vector versus value approximations for $\frac{n_c}{n} = 0.2$.



Vector versus value approximations for $\frac{n_c}{n} = 0.1$.



Further Work

There are some obvious further steps which could be taken based on these results.

- Distribute the implementation and obtain timing results.
- Test on a wider set of graphs, for a more meaningful statistical analysis.
- Test methods on truly massive graphs.

Summary and Thanks

- We developed a "spectrally aware" coarsening method with impressive empirical eigenvalue preservation properties, particularly on meshes.
- We further find that this method is not particularly good at preserving exact eigenvector embeddings.
- It is empirically suggested that none of the coarsening methods in this presentation can simultaneously preserve eigenvectors and eigenvalues of the original graph.

Thank you!

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[2] Jie Chen and Ilya Safro. "Algebraic distance on graphs". In: *SIAM Journal on Scientific Computing* 33.6 (2011), pp. 3468–3490.

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[6] George Karypis and Vipin Kumar. "A fast and high quality multilevel scheme for partitioning irregular graphs". In: *SIAM Journal on scientific Computing* 20.1 (1998), pp. 359–392.

[7] George Karypis and Vipin Kumar. "METIS: A software package for partitioning unstructured graphs, partitioning meshes, and computing fill-reducing orderings of sparse matrices". In: (1997).

[8] Tuhin Sahai, Alberto Speranzon, and Andrzej Banaszuk. "Hearing the clusters of a graph: A distributed algorithm". In: *Automatica* 48.1 (2012), pp. 15–24.

[9] Hongyu Zhu, Stefan Klus, and Tuhin Sahai. "A Dynamic Mode Decomposition Approach for Decentralized Spectral Clustering of Graphs". In: *arXiv preprint arXiv:2203.00004* (2022).